Exercice 1 : *(7 points)*

1.

$$\begin{aligned} z_1 &= 3 - 2i(4+i) \\ &= 3 - 8i - 2i^2 \\ &= 3 - 8i + 2 \\ &= 5 - 8i \end{aligned}$$

/1 point

2.

$$\begin{aligned} z_2 &= (3+2i)^2 \\ &= 3^2 + 2 \times 2i \times 3 + (2i)^2 \\ &= 9 + 12i + 4i^2 \\ &= 5 + 12i \end{aligned}$$

/1 point

3.

$$\begin{aligned} z_3 &= (-1+i)(9-5i) \\ &= -9 + 5i + 9i - 5i^2 \\ &= -4 + 14i \end{aligned}$$

/1 point

4.

$$\begin{aligned} z_4 &= \frac{1}{5+7i} \\ &= \frac{5-7i}{(5-7i)(5+7i)} \\ &= \frac{5-7i}{5^2+7^2} \\ &= \frac{5}{74} - \frac{7}{74}i \end{aligned}$$

/1,5 point

5.

$$\begin{aligned} z_5 &= \frac{1+i}{2-8i} \\ &= \frac{(1+i)(2+8i)}{(2-8i)(2+8i)} \\ &= \frac{2+8i+2i+8i^2}{2^2+8^2} \\ &= \frac{-6+10i}{68} \\ &= -\frac{3}{34} + \frac{5}{34}i \end{aligned}$$

/1,5 point

6.

$$\begin{aligned} z_6 &= \overline{(5+i)(4-2i)} \\ &= \overline{(5+i)}\ \overline{(4-2i)} \\ &= (5-i)(4+2i) \\ &= 20 + 10i - 4i - 2i^2 \\ &= 22 + 6i \end{aligned}$$

/1,5 point

Exercice 2 : *(7 points)*

1.
$$\begin{aligned} 2iz = 1 + z &\iff 2iz - z = 1 \\ &\iff z(-1+2i) = 1 \\ &\iff z = \frac{1}{-1+2i} \\ &\iff z = \frac{-1-2i}{5} \\ &\iff z = -\frac{1}{5} - \frac{2}{5}i \end{aligned}$$

/2 points

2. On pose $z = a + ib$ avec $a \in \mathbb{R}$ et $b \in \mathbb{R}$

$$\begin{aligned} 3z - 5\overline{z} = 7i &\iff 3(a+ib) - 5(a-ib) = 7i \\ &\iff 3a + 3bi - 5a + 5bi = 7i \\ &\iff -2a + 8bi = 7i \\ &\iff \begin{cases} -2a = 0 \\ 8b = 7 \end{cases} \\ &\iff \begin{cases} a = 0 \\ b = \dfrac{7}{8} \end{cases} \\ &\iff z = \frac{7}{8}i \end{aligned}$$

/2 points

3. Soit Δ le discriminant de $7z^2 - z + 1 = 0$:

$$\begin{aligned} \Delta &= (-1)^2 - 4 \times 7 \times 1 \\ &= 1 - 28 \\ &= -27 \end{aligned}$$

$\Delta < 0$ donc $7z^2 - z + 1 = 0$ admet deux solutions complexes conjuguées :

$$\begin{aligned} z_1 &= \frac{1 - i\sqrt{27}}{14} \\ &= \frac{1}{14} - \frac{\sqrt{27}}{14}i \end{aligned} \quad \text{et} \quad \begin{aligned} z_2 &= \frac{1 + i\sqrt{27}}{14} \\ &= \frac{1}{14} + \frac{\sqrt{27}}{14}i \end{aligned}$$

/3 points

Exercice 3 : ***(9 points)***

Pour tout $z \in \mathbb{C}$, on a :

$$P(z) = z^4 - 6z^3 + 23z^2 - 34z + 26$$

1. Pour tout $z \in \mathbb{C}$, on a :

$$\begin{aligned}\overline{P(z)} &= \overline{z^4 - 6z^3 + 23z^2 - 34z + 26}\\ &= \overline{z^4} - \overline{6z^3} + \overline{23z^2} - \overline{34z} + \overline{26}\\ &= \overline{z^4} - \overline{6}\,\overline{z^3} + \overline{23}\,\overline{z^2} - \overline{34}\,\overline{z} + \overline{26}\\ &= \overline{z}^4 - 6\overline{z}^3 + 23\overline{z}^2 - 34\overline{z} + 26\\ &= P\left(\overline{z}\right)\end{aligned}$$

/1 point

Si $\alpha \in \mathbb{C}$ vérifie $P(\alpha) = 0$, alors on a :

$$\begin{aligned}P(\overline{\alpha}) &= \overline{P(\alpha)}\\ &= \overline{0}\\ &= 0\end{aligned}$$

Ainsi $\overline{\alpha}$ vérifie $P(\overline{\alpha}) = 0$. ***/1 point***

2. On a :

$$\begin{aligned}(1+i)^2 &= 1^2 + 2i + i^2\\ &= 2i\end{aligned}$$

On a :

$$\begin{aligned}P(1+i) &= (1+i)^4 - 6(1+i)^3 + 23(1+i)^2 - 34(1+i) + 26\\ &= (2i)^2 - 6 \times 2i(1+i) + 23 \times 2i - 34 - 34i + 26\\ &= -4 - 12i + 12 + 46i - 34 - 34i + 26\\ &= 0\end{aligned}$$

/2 points

On vient de démontrer que $P(1+i) = 0$ donc $1+i$ est solution de l'équation $P(z) = 0$.
De plus, d'après la question précédente, on en déduit que $P(\overline{1+i}) = 0$ c'est-à-dire $P(1-i) = 0$ et donc $1-i$ est également solution de $P(z) = 0$. ***/1 point***

3. Pour tout $z \in \mathbb{C}$, on a :

$$\begin{aligned}P(z) &= \big(z-(1+i)\big)\big(z-(1-i)\big)\left(z^2 - 4z + 13\right)\\ &= \left(z^2 - (1+i)z - (1-i)z + (1+i)(1-i)\right)\left(z^2 - 4z + 13\right)\\ &= \left(z^2 - 2z + 2\right)\left(z^2 - 4z + 13\right)\\ &= z^4 - 4z^3 + 13z^2 - 2z^3 + 8z^2 - 26z + 2z^2 - 8z + 26\\ &= z^4 - 6z^3 + 23z^2 - 34z + 26\end{aligned}$$

/2 points

4. On a :

$$\begin{aligned}P(z) = 0 &\iff \big(z-(1+i)\big)\big(z-(1-i)\big)\left(z^2 - 4z + 13\right)\\ &\iff z-(1+i) = 0 \quad \text{ou} \quad z-(1-i) = 0 \quad \text{ou} \quad z^2 - 4z + 13 = 0\\ &\iff z = 1+i \quad \text{ou} \quad z = 1-i \quad \text{ou} \quad z^2 - 4z + 13 = 0\end{aligned}$$

4. *(Suite)* Résolvons $z^2 - 4z + 13 = 0$. Soit Δ le discriminant de $z^2 - 4z + 13 = 0$:

$$\begin{aligned}\Delta &= (-4)^2 - 4 \times 1 \times 13 \\ &= 16 - 52 \\ &= -36\end{aligned}$$

$\Delta < 0$ donc $z^2 - 4z + 13 = 0$ admet deux solutions complexes conjuguées :

$$\begin{aligned}z_1 &= \frac{4 - i\sqrt{36}}{2} \\ &= 2 - 3i\end{aligned} \qquad \text{et} \qquad \begin{aligned}z_2 &= \frac{4 + i\sqrt{36}}{2} \\ &= 2 + 3i\end{aligned}$$

Finalement,

$$P(z) = 0 \iff z = 1 + i \quad \text{ou} \quad z = 1 - i \quad \text{ou} \quad z = 2 - 3i \quad \text{ou} \quad z = 2 + 3i$$

/2 points